Double integrals and changing the order of integration 4

Given the following double integral:

$$\iint_D xy\,dx\,dy$$

$$D = \{(x, y) \mid 1 \leq y \leq 4 \wedge 0 \leq x \leq y^2\}$$

1. Calculate the value of the given integral in the specified region.
2. Change the order of integration. Integrate first with respect to the variable y and then with respect to the variable x.

Solution

1. Setting up the integral:

$$\int_1^4 \int_0^{y^2} xy\,dx\,dy$$

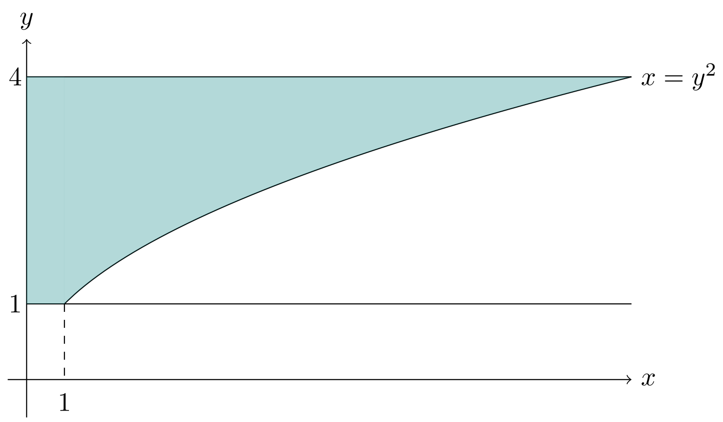
Solving the inner integral:

$$\int_0^{y^2} xy\,dx = \left.\frac{x^2 y}{2}\right|_0^{y^2} = \frac{y^5}{2}$$

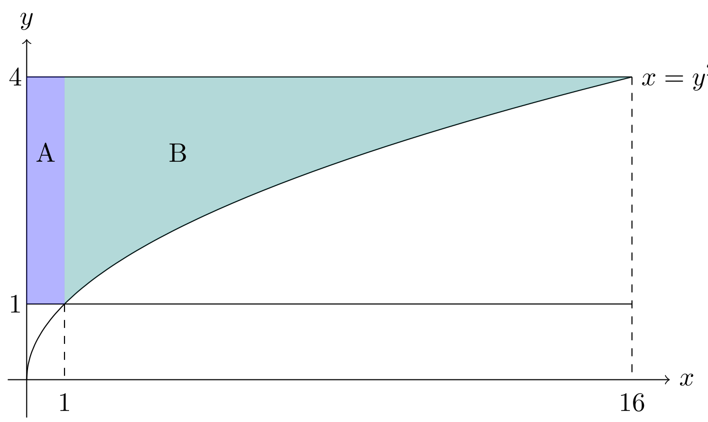
Solving the outer integral:

$$\int_1^4 \left(\frac{y^5}{2}\right) dy = \left.\frac{y^6}{12}\right|_1^4 = \frac{1365}{4}$$

2. Before solving, let's graph the area of the region:



If we calculate by changing the order of integration, now the expression has two different "layers," one that goes from 0 to 1 and another that goes from 1 to 16, and therefore two areas need to be calculated:



We calculate Area A: The first refers to the region bounded by 0 and 1 on the x-axis and $y = 1$ to $y = 4$ on the y-axis:

$$\int_0^1 \int_1^4 xy\,dy\,dx$$

The result of the first integral:

$$xy^2/2$$

Evaluating at the limits, the result is $15/2x$. Solving the second integral:

$$\int_0^1 15/2x\,dx = (15/2)x^2/2$$

Evaluating at the limits, the result is $15/4$.

We calculate Area B: Now we proceed with the second double integral that considers the region bounded by 1 and 16 on the x-axis. On the y-axis, we have $y = \sqrt{x}$, evaluating at $x = 16$: $y = 4$:

$$\int_1^{16}\int_{\sqrt{x}}^4 xy\,dy\,dx$$

Solving the first integral: $xy^2/2$. Evaluating at the limits: $x8 - x(\sqrt{x})^2/2 = x8 - x^2/2$.
Solving the second integral:

$$\int_1^{16} x8 - x^2/2 = (x^2)8/2 - x^3/3$$

Evaluating at the limits: $675/2$. Finally, we add the two calculated areas: $675/2 + 15/4 = 1365/4$.